On the minimum rank of a graph

Jisu Jeong

June 21, 2013

Definition

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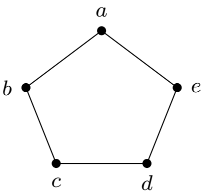

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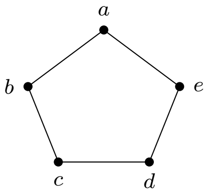

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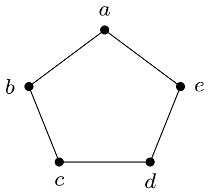

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Motivation

Some properties

Some properties

- The miminum rank of G is at most 1 if and only if G can be expressed as the union of a clique and an independent set.
- A path G is the only graph of minimum rank $|V(G)| - 1$.
- If G' is an induced subgraph of G, then $\mathrm{mr}(G') \leq \mathrm{mr}(G)$.

Main topics

- The minimum rank of a random graph over the binary field. (joint work with Choongbum Lee, Po-Shen Loh, and Sang-il Oum)
- An algorithm to decide that the input graph has the minimum rank at most k over $\mathbb{F}_q$, for a fixed integer k. (joint work with Sang-il Oum)

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Known results

The minimum rank of a random graph over a field.

	$\mathbb{R}^{\dagger}$	${\mathbb{F}_2}^{\ddagger}$
$G(n,1/2)$	$0.147n < \mathrm{mr} < 0.5n$	$n-\sqrt{2n} \le \mathrm{mr}$
$G(n,p)$	$cn < \mathrm{mr} < dn$	

† Hall, Hogben, Martin, and Shader, 2010
‡ Friedland and Loewy, 2010

Our results

Let $p(n)$ be a function s.t. $0 < p(n) \leq \frac{1}{2}$ and $np(n)$ is increasing.
We prove that the minimum rank of $G(n, 1/2)$ and $G(n, p(n))$ over the binary field is at least $n - o(n)$ a.a.s.
We have two different proofs.

Theorem

- $\mathrm{mr}(\mathbb{F}_2, G(n, 1/2)) \geq n - 1.415\sqrt{n}$ a.a.s.
- $\mathrm{mr}(\mathbb{F}_2, G(n, p(n))) \geq n - 1.178\sqrt{n/p(n)}$ a.a.s.

Theorem

- $\mathrm{mr}(\mathbb{F}_2, G(n, 1/2)) \geq n - \sqrt{2n} - 1.1$ a.a.s.
- $\mathrm{mr}(\mathbb{F}_2, G(n, p(n))) \geq n - 1.483\sqrt{n/p(n)}$ a.a.s.

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Known results

Theorem(Berman, Friedland, Hogben, Rothblum, and Shader, 08)

The computation of the minimum rank over $\mathbb{R}$ and $\mathbb{C}$ is decidable.

Theorem(Ding and Kotlov, 06)

For every nonnegative integer k, the set of graphs of minimum rank at most k is characterized by finitely many forbidden induced subgraphs, each having at most $(\frac{q^k+2}{2})^2$ vertices.

Our results

Theorem

Let k be a fixed positive integer and $\mathbb{F}_q$ be a fixed finite field. There exists an $O(|V(G)|^2)$-time algorithm that decides whether the input graph G has the minimum rank over $\mathbb{F}_q$ at most k.

Proofs

- Monadic second-order logic and Courcelle's thm
- Dynamic programming
- Kernelization

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Proofs

- Monadic second-order logic ($\exists$, $\forall$, $\vee$, $\wedge$, $\in$, $\sim$)
  - $\mathrm{mr}(\mathbb{F}_2, G) \leq k$
  - $\mathrm{mr}(\mathbb{F}_q, G) \leq k \to H$ is an induced subgraph of G
- Courcelle's thm
  - MS formula can be decided in linear time if the input graph is given with its p-expression.

Our results

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Proofs

- Dynamic programming
  - The number of partial solutions are bounded if an input graph has the minimum rank at most k.
  - H is an induced subgraph of G.

Our results

Theorem

Let k be a fixed positive integer and $\mathbb{F}_q$ be a fixed finite field. There exists an $O(|V(G)|^4)$-time algorithm that decides whether the input graph G has the minimum rank over $\mathbb{F}_q$ at most k.

Proofs

- Kernelization
  - If $|V(G)| > (\frac{q^k+2}{2})^2$, find a vertex v such that $\mathrm{mr}(\mathbb{F}_q, G) \leq k \Leftrightarrow \mathrm{mr}(\mathbb{F}_q, G \setminus v) \leq k$.

Future work

- It is still unknown whether the minimum rank can be computed in polynomial time.
- The lower bound for $G(n, p(n))$ has a possibility of being improved. (1.483)

Theorem

- $\mathrm{mr}(\mathbb{F}_2, G(n, 1/2)) \geq n - \sqrt{2n} - 1.1$ a.a.s.
- $\mathrm{mr}(\mathbb{F}_2, G(n, p(n))) \geq n - 1.483\sqrt{n/p(n)}$ a.a.s.

Future work

- A nontrivial upper bound of the expectation of the minimum rank of a random graph over the binary field is an open question.
- The minimum rank of a random graph over the other fields is unknown.

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Thank you.

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Let $p(n)$ be a function s.t. $0 < p(n) \leq \frac{1}{2}$ and $np(n)$ is increasing.
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Theorem

- $\mathrm{mr}(\mathbb{F}_2, G(n, 1/2)) \geq n - \sqrt{2n} - 1.1$ a.a.s. (Proof)
- $\mathrm{mr}(\mathbb{F}_2, G(n, p(n))) \geq n - 1.483\sqrt{n/p(n)}$ a.a.s.

Sketch of the proof

Theorem

Let $\mathbb{F}_2$ be the binary field and $G(n, \frac{1}{2})$ be a random graph. Then,

$$\operatorname{mr}\left(\mathbb{F}_2, G(n, \frac{1}{2})\right) \geq n - \sqrt{2n} - 1.1$$

asymptotically almost surely.

Sketch of the proof.

$G = G(n, 1/2)$

$\mathcal{G}_n$: a set of all graphs with a vertex set $\{1, 2, \cdots, n\}$ $S_n(\mathbb{F}_2)$: a set of all $n \times n$ symmetric matrices over the binary field

There can be many different matrices representing the same graph. If one of them has rank less than r, then the minimum rank of this graph is less than r. Thus,

$$\sum_{\substack{\mathrm{mr}(\mathbb{F}_2,H)<r \\ H\in\mathcal{G}_2}} \mathbb{P}[G=H] \le \sum_{\substack{\mathrm{rank}(N)<r \\ N\in\mathcal{M}}} \mathbb{P}[G=G(N)].$$

Let M be an $n\times n$ random symmetric matrix s.t. every entry in the upper triangle and diagonal of M is 1 with $1/2$. For $N\in S_n(\mathbb{F}_2)$, we have

$$\mathbb{P}[G=G(N)] = 2^n\mathbb{P}[M=N]$$

because the diagonal entries are decided with probability $1/2$ independently at random.

Therefore, we have

$$\begin{aligned}\mathbb{P}[\mathrm{mr}(\mathbb{F}_2, G) < n - L_n] &= \sum_{\substack{\mathrm{mr}(\mathbb{F}_2,H)<n-L_n \\ H\in\mathcal{G}}} \mathbb{P}[G = H] \\ &\leq \sum_{\substack{\mathrm{rank}(N)<n-L_n \\ N\in\mathcal{M}}} \mathbb{P}[G = G(N)] \\ &= 2^n \sum_{\substack{\mathrm{rank}(N)<n-L_n \\ N\in\mathcal{M}}} \mathbb{P}[M = N] \\ &= 2^n \mathbb{P}[\mathrm{rank}(M) < n - L_n] \\ &= 2^n \mathbb{P}[\mathrm{nullity}(M) > L_n].\end{aligned}$$

It is enough to show that $\mathbb{P}[\mathrm{nullity}(M) > \sqrt{2n} + 1.1]$ is $o(1/2^n)$.
So, we focus on $\mathbb{P}[\mathrm{nullity}(M) = L_n]$.

Lemma

Let M_i be an $i \times i$ random symmetric matrix such that every entry in the upper triangle and diagonal of M_i is 1 with probability $\frac{1}{2}$ independently at random. And let $P_{i,k}$ be the probability that M_i has nullity k. Then, $P_{1,0} = P_{1,1} = P_{2,0} = \frac{1}{2}$, $P_{2,1} = \frac{3}{8}$, $P_{2,2} = \frac{1}{8}$, $P_{i,-1} = 0$ for all i, $P_{i,k} = 0$ for all $i < k$, and

$$P_{i,k} = \frac{1}{2}P_{i-1,k} + \frac{1}{2^i}P_{i-1,k-1} + \frac{1}{2}(1 - \frac{1}{2^{i-1}})P_{i-2,k}$$

for $i \geq 3$, $k \geq 0$.